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# Exact Solutions for Coupled Einstein, Dirac, Maxwell, and Zero-Mass Scalar Fields

Amar Chandra Patra[1] and Dipankar Ray[2]

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Coupled equations for Einstein, Maxwell, Dirac, and zero-mass scalar fields studied by Krori, Bhattacharya, and Nandi are integrated for plane-symmetric time-independent case. It is shown that solutions do not exist for the plane-symmetric time-dependent case.

## 1. INTRODUCTION

In a recent paper, Krori *et al.* (1983) reduced the field equations for Einstein-Maxwell-Dirac zero-mass scalar fields for time-independent and time-dependent cases to two sets of coupled differential equations. They gave some particular solutions for the time-independent case and indicated how some solutions for the time-dependent case could be found. In the present paper the coupled equations for both the time-independent and time-dependent cases are integrated.

## 2. FIELD EQUATIONS

The field equations of the Einstein-Maxwell-Dirac-massless scalar field are

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-8\pi(E_{\mu\nu}+S_{\mu\nu}+T_{\mu\nu}) \tag{2.1}$$

$$F^{\alpha\beta}_{;\beta}=0 \tag{2.2}$$

$$R_{\alpha\beta;\nu}+F_{\beta\nu;\beta}+F_{\nu\alpha;\beta}=0 \tag{2.3}$$

$$\gamma^{\mu}\nabla_{\mu}\psi=0 \tag{2.4}$$

$$g^{\mu\nu}\phi_{;\mu\nu}=0 \tag{2.5}$$

[1]Department of Mathematics, Darjeeling Government College, Darjeeling 734101, West Bengal, India.
[2]Department of Mathematics, University of Jadavpur, Calcutta 700032, India.



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where the energy-momentum tensors for electromagnetic, Dirac, and scalar fields are, respectively,

$$E_{\mu\nu} = -F_{\mu\alpha}F^{\alpha}_{\nu} + \tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{2.6}$$

$$T_{\mu\nu} = \tfrac{1}{4}[\psi^{+}\gamma_{\mu}\nabla_{\nu}\psi + \psi^{+}\gamma_{\nu}\nabla_{\mu}\psi - (\nabla_{\mu}\psi^{+})\gamma_{\nu}\psi - (\nabla_{\nu}\psi^{+})\gamma_{\mu}\psi] \tag{2.7}$$

$$S_{\mu\nu} = \phi_{;\mu}\phi_{,\nu} - \tfrac{1}{2}g_{\mu\nu}(g^{1m}\phi_{,1}\phi_{,m}) \tag{2.8}$$

We use units in which $h = c = 1$. We adopt the conventions of Jauch and Rohrlich (1976) for Dirac $\gamma$ matrices and notations of Brill and Wheeler (1957) with regard to $\psi^{+}$, $\psi^{*}$, and $\nabla_{\mu}\psi$.

Krori *et al.* (1983) considered the plane-symmetric line element

$$ds^2 = e^{2u}(dt^2 - dx^2) - e^{2v}(dy^2 + dz^2) \tag{2.9}$$

where $u$ and $v$ are functions of $x$ alone for both time-independent and time-dependent Dirac field.

## 3. TIME-INDEPENDENT DIRAC FIELD

### 3.1. Equations

When the Dirac field $\psi$ is time-independent, equations (2.4) and (2.9) give

$$\psi = e^{-(v+u/2)}\psi_0 \tag{3.1}$$

where $\psi_0$ is a constant spinor.

The nonvanishing components of $T_{\mu\nu}$ are

$$T_{20} = \tfrac{1}{4}e^{-u}(v_{,1} - u_{,1})\psi^{+}\gamma^{1}\gamma^{2}\gamma^{0}\psi \tag{3.2}$$

$$T_{30} = \tfrac{1}{4}e^{-u}(v_{,1} - u_{,1})\psi^{+}\gamma^{1}\gamma^{2}\gamma^{0}\psi \tag{3.3}$$

where a comma denotes differentiation with respect to $x$.

Since $R_{20} = R_{30} = 0$, this implies that

$$T_{20} = T_{30} = 0 \tag{3.4}$$

Equations (3.2)-(3.4) give

$$\psi_0 = \alpha_0 \begin{pmatrix} 1 \\ \pm 1 \\ i \\ \pm i \end{pmatrix} \tag{3.5}$$

where $\alpha_0$ is a constant.

Thus, $\psi$ is obtained from equation (3.1) when $u$ and $v$ are the ones appearing in (2.9).

Equations (2.2)-(2.4) give the electromagnetic field,

$$F_{01} = c_1 e^{-2v}, \qquad F_{23} = c_2 e^{-2v} \tag{3.6}$$

where $c_1$ and $c_2$ are constants.

With the help of (2.9), equation (2.1) reduces to

$$v_{,1}^2 + 2u_{,1}v_{,1} = a e^{2u-4v} + b e^{-4v} \tag{3.7}$$

$$2v_{,11} - 2u_{,1}v_{,1} + 3v_{,1}^2 = a e^{2u-4v} - b e^{-4v} \tag{3.8}$$

$$u_{,11} + v_{,11} + v_{,1}^2 = -a e^{2u-4v} - b e^{-4v} \tag{3.9}$$

Here

$$a = -4\pi(c_1^2 + c_2^2) \tag{3.10}$$

$$b = 4\pi d^2 \tag{3.11}$$

where $d$ is a constant.

Krori *et al.* (1983) give some particular solutions of equations (3.7)-(3.9). We present here the general solutions of the same equations. Once $u$ and $v$ are obtained, $\psi$ can be obtained from (3.1) and (3.5).

### 3.2. Solutions

Since $u$ and $v$ are functions of $x$ alone, we can take

$$u = u(v)$$

Therefore

$$u_{,1} = u_v v_{,1}$$

$$u_{,11} = u_{vv} v_{,1}^2 + u_v v_{,11}$$

Then one can reduce equations (3.7)-(3.9) to

$$(2u_v + 1)v_{,1}^2 = a e^{2u-4v} + b e^{-4v} \tag{3.12}$$

$$2v_{,11} + (3 - 2u_v)v_{,1}^2 = a e^{2u-4v} - b e^{-4v} \tag{3.13}$$

$$(u_v + 1)v_{,11} + (u_{vv} + 1)v_{,1}^2 = -a e^{2u-4v} - b e^{-4v} \tag{3.14}$$

To solve the coupled equations (3.12)-(3.14) one can proceed as follows: Eliminating $v_{,1}^2$ and $v_{,11}$ from (3.12)-(3.14), one gets

$$(a e^{2u} + b)u_{vv} + 2a e^{2u}u_v^2 + (3u_v + 1)a e^{2u} = 0 \tag{3.15}$$

Integrating (3.15), one obtains

$$u_v = \frac{(K^2 - b - a\,e^{2u}) \pm K(K^2 - b - a\,e^{2u})^{1/2}}{a\,e^{2u} + b} \tag{3.16}$$

where $K$ is a constant of integration.

Again, integrating (3.16), one easily gets

$$e^{2v} = \frac{K_1[(K^2 - b)^{1/2} + (K^2 - b - a\,e^{2u})^{1/2}]^g}{(a\,e^{2u})^{g+1}} \tag{3.17}$$

where $K_1$ is a constant of integration and

$$g = \pm 2K/(K^2 - b)^{1/2}$$

Inserting the value of $v$ from (3.17) into equation (3.12) and integrating, one obtains

$$\pm x + K_2 = \frac{K_1}{(2m)^{g+3}} \int \left(1 + \frac{1}{y}\right)^{g+2} dy \tag{3.18}$$

where

$$y = \frac{[m - (m^2 - a\,e^{2u})^{1/2}]^2}{a\,e^{2u}} \tag{3.19}$$

$K_2$ is a constant of integration and $m^2 = K^2 - b$.

Putting (3.17) and (3.18) into equations (3.12)-(3.14), one can check that all the equations are satisfied. Hence the complete set of solutions of equations (3.7)-(3.9) is given by (3.17) and (3.18).

Note that (3.17) and (3.18) can also be obtained from equations (3.12) and (3.13) only. Thus, equation (3.14) is really superfluous.

## 4. TIME-DEPENDENT DIRAC FIELD

### 4.1. Equations

We assume that the Dirac field $\psi$ is a function of $x$ and $t$ and without any loss of generality we choose

$$\psi = \psi_0(x)\, e^{-i\omega t} \tag{4.1}$$

where $\psi_0(x)$ is a spinor and $\omega$ is a real constant.

Equations (2.4), (2.9), and (4.1) give

$$\psi = \exp[-(v + u/2)](\cos \omega x + i\gamma^1\gamma^0 \sin \omega x) \times [\exp(-i\omega t)]\psi_c \tag{4.2}$$

where $\psi_c$ is an arbitrary constant spinor.

The nonvanishing components of $T_{\mu\nu}$ are

$$T_{00}=T_{11}=\tfrac{1}{4}e^{-u}\psi^{+}(4i\omega\gamma^{0})\psi \tag{4.3}$$

$$T_{10}=T_{01}=\tfrac{1}{4}e^{-u}\psi^{+}(-4i\omega\gamma^{1})\psi \tag{4.4}$$

$$T_{20}=T_{02}=\tfrac{1}{4}e^{-u}\psi^{+}[-2i\omega\gamma^{2}+\gamma^{1}\gamma^{2}\gamma^{0}(v_{,1}-u_{,1})]\psi \tag{4.5}$$

$$T_{30}=T_{03}=\tfrac{1}{4}e^{-u}\psi^{+}[-2i\omega\gamma^{3}+\gamma^{1}\gamma^{3}\gamma^{0}(v_{,1}-u_{,1})]\psi \tag{4.6}$$

Since $R_{01}=R_{02}=R_{03}=0$, this implies

$$T_{01}=T_{02}=T_{03}=0 \tag{4.7}$$

Equations (4.4)-(4.7) give

$$\psi_c=\begin{pmatrix} s \\ \pm s \\ q \\ \pm q \end{pmatrix} e^{1\lambda} \tag{4.8}$$

where $s$, $q$, and $\lambda$ are real constants.

Thus, $\psi$ is obtained from (4.2) when $u$ and $v$ are the ones appearing in (2.9).

In this case the field equations are

$$v_{,1}^{2}+2u_{,1}v_{,1}=a\,e^{2u-4v}+b\,e^{-4v}-8\pi\,e^{2u}T_{11} \tag{4.9}$$

$$2v_{,11}-2u_{,1}v_{,1}+3v_{,1}^{2}=a\,e^{2u-4v}-b\,e^{-4v}+8\pi T_{00}\,e^{2u} \tag{4.10}$$

$$u_{,11}+v_{,11}+v_{,1}^{2}=-a\,e^{2u-4v}-b\,e^{-4v} \tag{4.11}$$

We now seek the solutions of equations (4.9)-(4.11). Such solutions, if obtained, will give $\psi$ from (3.1) and (3.5).

## 4.2. Solutions

From (4.2), one can easily obtain

$$\psi=e^{-(v+u/2)-i\omega t+i\lambda}\left[\cos\omega x\begin{pmatrix} s \\ \pm s \\ q \\ \pm q \end{pmatrix}+\sin\omega x\begin{pmatrix} \pm q \\ q \\ \pm s \\ s \end{pmatrix}\right]$$

$$\psi^{+}=e^{-(v+u/2)-i\omega t+i\lambda}[(s\pm s\;q\pm q)\cos\omega x+(\pm q\;q\pm s\;s)\sin\omega x]$$

$$(\psi^{+})^{*}=e^{-(v+u/2)+i\omega t-i\lambda}[(s\pm s\;q\pm q)\cos\omega x+(\pm q\;q\pm s\pm s)\sin\omega x]$$

Hence from (4.3), one can get

$$T_{00}=T_{11}=2i\omega(s^2-q^2)\,e^{-2(u+v)}\cos 2\omega x \tag{4.12}$$

Inserting the value of $T_{00}=T_{11}$ from (4.12) into equations (4.9)-(4.11), one obtains

$$v_{,1}^2+2u_{,1}v_{,1}=a\,e^{2u-4v}+b\,e^{-4v}+A\,e^{-2v}\cos 2\omega x \tag{4.13}$$

$$2v_{,11}-2u_{,1}v_{,1}+3v_{,1}^2=a\,e^{2u-4b}-b\,e^{-4v}-A\,e^{-2v}\cos 2\omega x \tag{4.14}$$

$$u_{,11}+v_{,11}+v_{,1}^2=-a\,e^{2u-4v}-b\,e^{-4v} \tag{4.15}$$

where

$$A=16\pi i\omega(q^2-s^2) \tag{4.16}$$

Subtracting (4.13) from (4.14), one obtains

$$v_{,11}+v_{,1}^2-2u_{,1}v_{,1}=-b\,e^{-4v}-A\,e^{-2v}\cos 2\omega x \tag{4.17}$$

Also adding (4.13) to (4.14), one gets

$$v_{,11}+2v_{,1}^2=a\,e^{2u-4v} \tag{4.18}$$

It was noted by Krori *et al.* (1983) that equations (4.15) and (4.18) together are equivalent to equations (3.7)-(3.9) obtained for the time-independent case. However, it is obvious that equations (4.15) and (4.18) are necessary but not sufficient for the coupled equations (4.13)-(4.15) to be satisfied.

In fact, it can be shown that the coupled equations (4.13)-(4.15) cannot be satisfied unless either $a=0$ or $A=0$ (proof is given in the Appendix).

We note that in view of equations (3.6) and (3.10), $a=0$ means the absence of the Maxwell field and in view of (4.16), $A=0$ means the absence of the Dirac field. Therefore, there is no solution of the Einstein-Maxwell-Dirac zero-mass scalar equations for the case under consideration.

## 5. CONCLUSION

In summary, all the time-independent solutions of Einstein-Maxwell-Dirac zero-mass scalar field equations, i.e., equations (2.1)-(2.8), that are of plane-symmetric form, i.e., of the form (2.9), are given by (3.17) and (3.18). Further, there is no plane-symmetric time-dependent solution of the Einstein-Maxwell-Dirac zero-mass scalar field except when either the Maxwell field or the Dirac field vanishes.

## APPENDIX

It will be shown that equations (4.13)-(4.15) admit solutions only if either $a=0$ or $A=0$

*Case 1.* Let $a=0$, $A \neq 0$. Then the solutions of equations (4.13)-(4.15) are given by

$$\begin{aligned} u &= (b/m^2 - \tfrac{1}{4}) \ln(mx+m_1) + m_2 x + m_3 \\ v &= \tfrac{1}{2} \ln(mx+m_1) \end{aligned} \tag{A1}$$

where $m$, $m_1$, $m_2$, and $m_3$ are constants and $A \cos 2\omega x = m m_2$.

*Case 2.* Let $a \neq 0$. Then equation (4.18) can be written as

$$e^{2u} = (1/a)(v_{,11} + 2v_{,1}^2)\, e^{4v} \tag{A2}$$

Differentiating (A2), we find

$$u_{,1} = \frac{v_{,111} + 4v_{,1}v_{,11}}{2(v_{,11}+2v_{,1}^2)} + 2v_{,1} \tag{A3}$$

and

$$u_{,11} = \left[\frac{v_{,111} + 4v_{,1}v_{,11}}{2(v_{,11}+2v_{,1}^2)}\right]_{,1} + 2v_{,11} \tag{A4}$$

Using (A2) and (A4) in (4.15), one gets

$$\left[\frac{v_{,111} + 4v_{,1}v_{,11}}{2(v_{,11}+2v_{,1}^2)}\right]_{,1} + 4v_{,11} + 3v_{,1}^2 = -b\, e^{-4v} \tag{A5}$$

Substituting the value of $u_{,1}$ from (A3) in (4.17) and simplifying, one gets

$$v_{,111}v_{,1} + 5v_{,1}^2 v_{,11} - v_{,11}^2 + 6v_{,1}^4 = (v_{,11}+2v_{,1}^2)(b\, e^{-4v} + A\, e^{-2v} \cos 2\omega x) \tag{A6}$$

Differentiating (4.17) and using (A3)-(A5), one obtains, after some calculation,

$$\begin{aligned} &v_{,111}v_{,1} + 5v_{,1}^2 v_{,11} - v_{,11}^2 + 6v_{,1}^4 \\ &\quad + \frac{v_{,11}+2v_{,1}^2}{2v_{,1}} [2bv_{,1}\, e^{-4v} + (b\, e^{-4v} + A\, e^{-2v} \cos 2\omega x)_{,1}] = 0 \end{aligned} \tag{A7}$$

Subtracting (A6) from (A7) and simplifying, one obtains

$$(v_{,11} + 2v_{,1}^2)\omega A\, e^{-2v} \sin 2\omega x = 0 \tag{A8}$$

Now, since $a \neq 0$, we see from (4.18) that $v_{,11} + 2v_{,1}^2 \neq 0$.

Thus, we observe from (A8) that the only possible case is $\omega = 0$, and $\omega = 0$ means $A = 0$, and consequently equations (4.13)-(4.15) reduce to equations (3.7)-(3.9) for the time-independent case, whose solutions are completely determined.

## REFERENCES

Brill, D., and Wheeler, J. A. (1957) *Reviews of Modern Physics* **29**, 465.

Jauch, J. M., and Rohrlich, F. (1976) *The Theory of Photons and Electrons*, Springer, New York, Appendix A2.

Krori, K. D., Bhattacharya, R., and Nandi, D. (1983). *Journal of Mathematical Physics*, **24**, 1574.